

General-relativistic rotation laws in rotating fluid bodies, new weak-field effects and the post-newtonian expansion

Patryk Mach, Edward Malec* and Michał Piróg

*M. Smoluchowski Institute of Physics, Jagiellonian University
Cracow, Poland*

** E-mail: Edward.Malec@uj.edu.pl
www.uj.edu.pl*

Recent general-relativistic extensions of Newtonian rotation laws for self-gravitating stationary fluids allow one to rederive, in the first post-Newtonian approximation, the well known geometric dragging of frames, and two new weak-field effects within rotating tori. These are the recently discovered anti-dragging and a new effect that measures the deviation from the Keplerian motion and/or the contribution of the fluids selfgravity. They can be applied to the study of the existence of the (post-)Newtonian limits of solutions and in investigations of inequalities relating parameters of rotating black holes.

Keywords: general relativistic hydrodynamics, rotation laws, gravitational effects, post-Newtonian approximation

1. Introduction

Stationary Newtonian hydrodynamic configurations — axially symmetric and self-gravitating — that rotate around a fixed axis, possess two peculiarities. This is a free-boundary problem — its domain has to be found simultaneously with the solution itself. In addition, the problem is undetermined — traditionally one fixes it by (almost) freely prescribing the so-called rotation curve, the angular velocities Ω_0 of the particles of a fluid. There exists an integrability condition — that Ω_0 and the angular momentum per unit mass j can be functions of a single variable r , where r is the distance from the rotation axis. The additional “stability” restriction $\frac{dj}{dr} \geq 0$ of Ref. 1 constrains freedom to choose the angular velocity rather weakly.

General-relativistic counterparts of these systems are also undetermined, but the way to specify the system is through prescribing the function $j(\Omega)$ — the angular momentum as a function of the angular velocity. This condition can be rephrased in principle in terms of geometric coordinates, but only through solving a nonlinear equation that defines the linear velocity. This vague remark shall be explained later; we want only to say that it is nontrivial to find an admissible form of $j(\Omega)$. The only known rotation law in general-relativistic hydrodynamics had been for a long time that proposed by Bardeen and Wagoner², with j being a linear function of the angular velocity. Galeazzi, Yoshida and Eriguchi³ have found recently a nonlinear angular velocity profile. None of these tends in the Newtonian limit into the Newtonian monomial rotation curves $\Omega_0 = w/r^\lambda$. In this talk we shall introduce general-relativistic rotation curves $j = j(\Omega)$ that in the nonrelativistic limit exactly coincide with $\Omega_0 = w/r^\lambda$ ($0 \leq \lambda \leq 2$; λ and w are constants). They comprise, in particular, the general-relativistic Keplerian rotation law that possesses the first post-Newtonian limit (1PN). The well known Keplerian massless disk of

dust satisfies our rotation law exactly in the Schwarzschild spacetime.

We shall consider fluids with the polytropic equation of state $p(\rho, S) = K(S)\rho^\gamma$, where S is the specific entropy of fluid. Then one has specific enthalpy $h(\rho, S) = K(S)\frac{\gamma}{\gamma-1}\rho^{\gamma-1}$. The entropy is assumed to be constant.

2. Rotation laws in Newtonian hydrodynamics

The Euler equations read

$$-\Omega_0^2 r = -\partial_r(U_0 + h), \quad 0 = -\partial_z(U_0 + h); \quad (1)$$

One easily notices that this system of equations, supplemented by the Poisson equation $\Delta U_0 = 4\pi G\rho$, is closed only if Ω_0 is known. Differentiation of the two equations in (1) with respect to z and r , respectively, and subtraction of the results yields $\partial_z \Omega_0 = 0$. The angular velocity depends only on r , the distance from a rotation axis: $\Omega_0 = \Omega_0(r)$. This constitutes the integrability condition.

3. Equations of general-relativistic hydrodynamics

We assume the axially symmetric *stationary* metric

$$ds^2 = -e^{\frac{2\nu}{c^2}}(dx^0)^2 + r^2 e^{\frac{2\beta}{c^2}} \left(d\phi - \frac{\omega}{c^3} dx^0 \right)^2 + e^{\frac{2\alpha}{c^2}} (dr^2 + dz^2). \quad (2)$$

Here $x^0 = ct$ is the rescaled time coordinate (c is the speed of light in vacuum), and r, z, ϕ are cylindrical coordinates. The metric potentials ν, β, ω and α depend on r and z only.

The Einstein equations, with the signature of the metric $(-, +, +, +)$, read $R_{\mu\nu} - g_{\mu\nu} \frac{R}{2} = 8\pi \frac{G}{c^4} T_{\mu\nu}$, where $T_{\mu\nu}$ is the stress-momentum tensor. We take the stress-momentum tensor $T^{\alpha\beta} = \rho(c^2 + h)u^\alpha u^\beta + pg^{\alpha\beta}$, where ρ is the baryonic rest-mass density, h is the specific enthalpy, and p is the pressure. The 4-velocity u^α is normalized, $g_{\alpha\beta}u^\alpha u^\beta = -1$. The coordinate (angular) velocity reads $\vec{v} = \Omega \partial_\phi$, where $\Omega = u^\phi/u^t$.

Define the linear velocity with respect to the locally nonrotating system of reference

$$V = r \left(\Omega - \frac{\omega}{c^2} \right) e^{(\beta-\nu)/c^2}. \quad (3)$$

The potentials α, β, ν , and ω satisfy equations that have been found by Komatsu, Eriguchi and Hachisu⁴. Here we recall a version similar to that used in Ref. 5. The relevant equations read

$$\begin{aligned} \Delta \nu &= 4\pi \frac{G}{c^2} e^{2\alpha/c^2} \left[\rho(c^2 + h) \frac{1 + V^2/c^2}{1 - V^2/c^2} + 2p \right] + \frac{1}{2c^4} r^2 e^{2(\beta-\nu)/c^2} \nabla \omega \cdot \nabla \omega \\ &\quad - \frac{1}{c^2} \nabla(\beta + \nu) \cdot \nabla \nu \\ \left(\Delta + \frac{2}{r} \partial_r \right) \omega &= -16\pi \frac{G}{c^2} e^{2\alpha/c^2} \rho(c^2 + h) \frac{\Omega - \omega/c^2}{1 - V^2/c^2} + \frac{1}{c^2} \nabla(\nu - 3\beta) \cdot \nabla \omega, \end{aligned}$$

where ∇ denotes the “flat” gradient operator. They constitute an overdetermined, but consistent, set of equations. We omit the remaining Einstein equations, since they yield corrections of higher orders. The general-relativistic Euler equations are integrable assuming that the angular momentum per unit mass,

$$j = u_\phi u^t = \frac{V^2}{\left(\Omega - \frac{\omega}{c^2}\right) \left(1 - \frac{V^2}{c^2}\right)}, \quad (4)$$

depends only on the angular velocity Ω ; $j \equiv j(\Omega)$. For a given $j(\Omega)$, one can insert Eq. (3) into Eq. (4) and find explicit dependence of the angular velocity on spatial coordinates. It is clear, that this depends crucially on the form of $j(\Omega)$.

Assuming $j \equiv j(\Omega)$ one gets a general-relativistic integro-algebraic Bernoulli equation, that embodies the hydrodynamic information carried by the continuity equations $\nabla_\mu T^{\mu\nu} = 0$ and the baryonic mass conservation $\nabla_\mu (\rho u^\mu) = 0$. It is given by $\ln\left(1 + \frac{h}{c^2}\right) + \frac{\nu}{c^2} + \frac{1}{2} \ln\left(1 - \frac{V^2}{c^2}\right) + \frac{1}{c^2} \int d\Omega j(\Omega) = C$.

4. Rotation laws in general-relativistic hydrodynamics

The general-relativistic rotation law employed in the literature Refs. 2, 4–7, has the form $j(\Omega) = A^2(\Omega_c - \Omega)$, where A and Ω_c are parameters. In the Newtonian limit and large A one arrives at the rigid rotation, $\Omega = \Omega_c$, while for small A one gets the constant angular momentum per unit mass. A three-parameter expression for j is proposed in Ref. 3. None of these rotation curves give in the Newtonian limit the monomial ones.

Two of us have found recently a new family of rotation laws⁸,

$$j(\Omega) \equiv \frac{w^{1-\delta} \Omega^\delta}{1 - \frac{\kappa}{c^2} w^{1-\delta} \Omega^{1+\delta} + \frac{\Psi}{c^2}}, \quad (5)$$

where $w, \delta, \kappa = (1 - 3\delta)/(1 + \delta) + \mathcal{O}(c^{-2})$ and Ψ are parameters.

It is notable that a massless disk made of dust rotating in Keplerian motion is an exact solution in the Schwarzschild spacetime. The general-relativistic Bernoulli equation acquires a simple algebraic form⁸, if $\delta \neq -1$:

$$\left(1 + \frac{h}{c^2}\right) e^{\nu/c^2} \sqrt{1 - \frac{V^2}{c^2}} \times \left(1 - \frac{\kappa}{c^2} w^{1-\delta} \Omega^{1+\delta} + \frac{\Psi}{c^2}\right)^{\frac{-1}{(1+\delta)\kappa}} = C. \quad (6)$$

The seemingly singular case $\delta = -1$, that corresponds to the constant linear velocity, is also described by the present formalism; one should take a limit $\delta \rightarrow -1$ in Eq. (5) and some of forthcoming formulae (Ref. 9).

Rotation curves $\Omega(r, z)$ ought to be recovered from the equation

$$\frac{w^{1-\delta} \Omega^\delta}{1 - \frac{\kappa}{c^2} w^{1-\delta} \Omega^{1+\delta} + \frac{\Psi}{c^2}} = \frac{V^2}{\left(\Omega - \frac{\omega}{c^2}\right) \left(1 - \frac{V^2}{c^2}\right)}.$$

In the Newtonian limit — the zeroth order of the post-Newtonian expansion (0PN) — one arrives at $\Omega_0 = w/r^{\frac{2}{1-\delta}}$.

The stability requirement of Ref. 1 imposes the condition $\delta \leq 0$. These two constants, w and δ , can be given a priori. Let us remark at this point that the rotation law (5), and consequently the Newtonian rotation $\Omega_0 = w/r^{2/(1-\delta)}$, applies primarily to single rotating toroids and toroids rotating around black holes. In the case of rotating stars one would have to construct a special differentially rotating law, with the aim to avoid singularity at the rotation axis.

The two limiting cases $\delta = 0$ and $\delta = -\infty$ correspond to the constant angular momentum per unit mass ($\Omega_0 = w/r^2$) and the rigid rotation ($\Omega = w$), respectively. The Keplerian rotation is given by $\delta = -1/3$ and $w^2 = GM$, where M is a mass (Ref. 8).

The 1PN approximation corresponds to the choice of metric exponents $\alpha = \beta = -\nu = -U$ with $|U| \ll c^2$. Define $\omega \equiv r^{-2}A_\phi$; it appears that A_ϕ satisfies $\Delta A_\phi - 2\frac{\partial_r A_\phi}{r} = -16\pi G r^2 \rho_0 \Omega_0$.

The spatial part of the metric

$$ds^2 = -\left(1 + \frac{2U}{c^2} + \frac{2U^2}{c^4}\right) (dx^0)^2 - 2c^{-3}A_\phi dx^0 d\phi + \left(1 - \frac{2U}{c^2}\right) (dr^2 + dz^2 + r^2 d\phi^2). \quad (7)$$

is conformally flat. Notice that in the Newtonian gauge imposed in (7) the geometric distance to the rotation axis is given by $\tilde{r} = r(1 - U_0/c^2) + \mathcal{O}(c^{-4})$. The relevant post-Newtonian approximation has been discussed in Ref. 8; we shall extract from therein information on the general-relativistic corrections to the angular velocity.

The angular velocity, up to the terms $\mathcal{O}(c^{-4})$ is given by

$$\Omega = \frac{w}{\tilde{r}^{2/(1-\delta)}} - \frac{2}{c^2(1-\delta)}\Omega_0 (U_0 + \Omega_0^2 r^2) + \frac{A_\phi}{r^2 c^2 (1-\delta)} - \frac{4}{c^2(1-\delta)}\Omega_0 h_0. \quad (8)$$

This expression can be reduced to $\Omega = \frac{w}{\tilde{r}^{2/(1-\delta)}} + \frac{A_\phi}{r^2 c^2 (1-\delta)} - \frac{4}{r c^2 (1-\delta)}\Omega_0 h_0$, in the case of test fluids, at the symmetry plane $z = 0$. For the massless dust, in the Schwarzschild geometry, we get $\Omega = \Omega_0 = \frac{w}{\tilde{r}^{3/2}}$.

The first term in (8) is simply the Newtonian rotation law rewritten as a function of the geometric distance from the rotation axis. The second term (denoted as $\Omega_{1,\text{NK}}$ in Fig.1) vanishes at the plane of symmetry, $z = 0$, for circular Keplerian motion of test fluids in the monopole potential $-GM/\sqrt{r^2 + z^2}$. It is sensitive both to the contribution of the disk self-gravity at the plane $z = 0$ and the deviation from the strictly Keplerian motion. The third term ($\Omega_{1,\text{geo}}$ in Figs 1 and 2) is responsible for the geometric frame dragging. The last term ($\Omega_{1,\text{dyn}}$ in Fig. 1) represents the recently discovered anti-dragging effect; it agrees (for the monomial angular velocities $\Omega_0 = wr^{-2/(1-\delta)}$) — with the result obtained earlier in Ref. 10.

In the following discussion we assume $w > 0$, which means $\Omega_0 > 0$, but the reasoning is symmetric under the parity operation $w \rightarrow -w$. The specific enthalpy $h \geq 0$ is nonnegative, thence $-\frac{4\Omega_0 h_0}{1-\delta}$ is nonpositive — the discovered in Ref. 10 instantaneous 1PN dynamic reaction slows the motion: it “anti-drags” a system.

In contrast to that, the well known geometric term with A_ϕ is positive¹⁰, and the contribution $\frac{A_\phi}{r^2(1-\delta)}$ to the angular velocity is positive — it pushes a rotating fluid body forward. Thus the last two terms in (8) counteract.

The specific enthalpy h_0 vanishes for dust, hence dust test bodies are exposed only to the frame dragging. For the rigid (uniform) rotation the correction terms are proportional to $1/(1-\delta)$ and they vanish, because $\delta = -\infty$.

Figure 1 displays contributions of each of the particular effects at the symmetry plane $z = 0$ of the disk; notice that the anti-dragging can prevail the geometric dragging. Figure 2 in turn demonstrates that the sum of all three contributing effects can be much larger than the geometric counterpart.

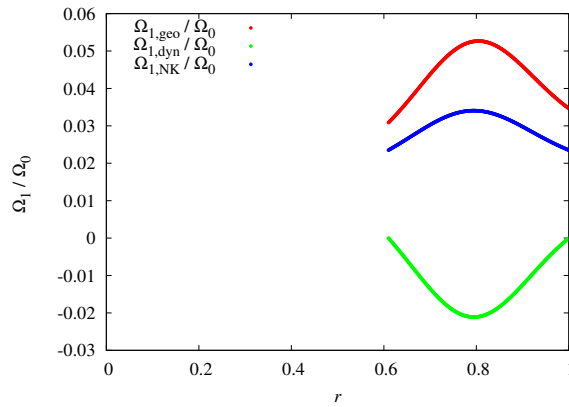


Fig. 1. The contributions of particular effects within the disk, $z = 0$.

5. Concluding remarks

We present general-relativistic rotation laws and derive a full form of the new weak-fields effects, including the recent dynamic anti-dragging effect of Ref. 10. The latter can be robust according to the numerics of Ref. 10, but the ultimate conclusion requires a fully general-relativistic treatment with the new rotation laws. It is conjectured that in some active galactic nuclei¹¹ the frame dragging can manifest through the Bardeen-Petterson effect¹². The two other effects may lead to its observable modifications in black hole systems with heavy disks. In the weak field approximation the angular velocity of toroids depends primarily on the distance from the rotation axis — as in the Newtonian hydrodynamics — but the weak fields contributions can make the rotation curve dependent on the height above the symmetry plane of rotation.

The new rotation laws would allow the investigation of self-gravitating fluid bodies in the regime of strong gravity for general-relativistic versions of Newtonian

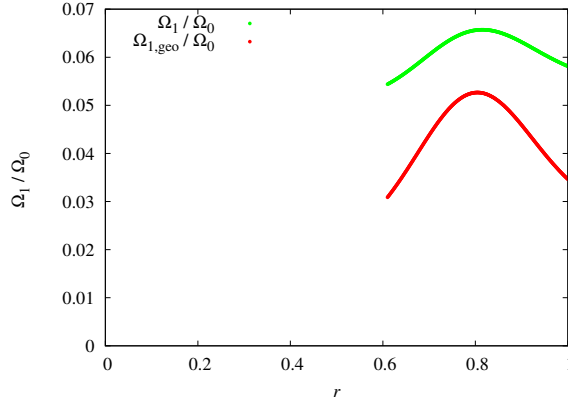


Fig. 2. The combined effect (the green line) and the geometric dragging (red line) within the disk, $z = 0$. Here $\Omega_1 = \Omega_{1,NK} + \Omega_{1,dyn} + \Omega_{1,geo}$

rotation curves. In particular, they can be used in order to describe stationary heavy disks in tight accretion systems with central black holes, for instance in products of the merger of compact binaries (pairs of black holes and neutron stars) (Refs. 13 and 14), but they might exist also in some active galactic nuclei.

These rotation laws can be applied to the study of various open problems in the post-Newtonian perturbation scheme of general-relativistic hydrodynamics — the investigation of convergence of the post-Newtonian perturbation scheme, as well as the existence of the Newtonian and post-Newtonian limits of solutions. They can be used to test the accuracy of recent inequalities relating the angular momentum, the mass and area of black holes^{15,16}.

References

1. J.-L. Tassoul, *Stellar Rotation* (CUP, Cambridge, UK, 2007).
2. J. M. Bardeen and R. Wagoner, **ApJ** **167**, 359 (1971).
3. F. Galeazzi, S. Yoshida and Y. Eriguchi, **AA** **541**, A156 (2012).
4. H. Komatsu, Y. Eriguchi, and I. Hachisu, **MNRAS** **237**, 355 (1989).
5. S. Nishida and Y. Eriguchi, **ApJ** **427**, 429 (1994).
6. E. Butterworth and I. Ipser, *Astrophys. J.* **200**, L103 (1975).
7. S. Nishida, Y. Eriguchi, and A. Lanza, **ApJ** **401**, 618 (1992).
8. P. Mach and E. Malec, *Phys. Rev.* **PRD** **91**, 124053 (2015).
9. J. Knopik, P. Mach and E. Malec, to appear in **APP** **B47**; arXiv:1509.01825 (2015).
10. P. Jaranowski, P. Mach, E. Malec and M. Piróg, **PRD** **91**, 024039 (2015).
11. J. M. Moran, *ASPC Conference Series*, **395**, 87M (2008).
12. J. Bardeen and J. Petterson, **ApJ** **195**, L65 (1975).
13. F. Pannarale, A. Tonita and L. Rezzolla, **ApJ** **727**, 95 (2011).

14. G. Lovelace et al., **CQG** **30**, 135004 (2013).
15. S. Dain, **PRL** **112**, 041101 (2014).
16. M. A. Khuri, JHEP. 06. 188 (2015).